JOURNAL OF APPROXIMATION THEORY 7, 167–185 (1973)

Convergence Rates of Certain Approximate Solutions to Fredholm Integral Equations of the First Kind*

GRACE WAHBA

Department of Statistics, University of Wisconsin, 1210 West Dayton Street, Madison Wisconsin 53706

Communicated by Oved Shisha

Received September 21, 1970

1. INTRODUCTION

We consider properties of certain approximate solutions of Fredholm integral equations of the first kind. Consider the equation

$$u(t) = \int_S K(t, s)\, z(s)\, ds, \qquad t \in T, \tag{1.1}$$

where S, T are closed, bounded intervals of the real line, $K(t, s)$ is a given kernel on $T \times S$ with appropriate properties, and $u(t)$ is known only for $t \in \Delta = \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < \cdots < t_n$, $[t_1, t_n] = T$.

Letting $u(t_i) = u_i$, we take as an approximate solution the function $\hat{z}$ which satisfies

$$u_i = \int_S K(t_i, s)\, \hat{z}(s)\, ds, \qquad i = 1, 2, ..., n \tag{1.2}$$

and minimizes a quadratic functional $J(z)$ of the form

$$J(z) = \| R^{-1/2} z \|_{\mathscr{L}_2}^2, \tag{1.3}$$

where $R^{-1/2}$ is a densely defined, unbounded linear operator on $\mathscr{L}_2(S)$ to be selected from a certain general class, and $\| \cdot \|_{\mathscr{L}_2}$ is the norm on $\mathscr{L}_2(S)$.

To define $R^{-1/2}$, let $R(s, s')$ be a continuous, symmetric positive definite kernel on $S \times S$. Then, by the theorems of Mercer, Hilbert, and Schmidt [8, pp. 242–246], the operator R, defined by

$$(Rf)(s) = \int_S R(s, u) f(u)\, du, \qquad f \in \mathscr{L}_2(S), \tag{1.4}$$

* This work was supported by the Mathematics Research Center, University of Wisconsin and by NSF Grant GA-1908 and Air Force Grant AFOSR-69-1803.

has an $\mathscr{L}_2(S)$-complete orthonormal system of eigenfunctions $\{\phi_\nu\}_{\nu=1}^\infty$ and corresponding eigenvalues $\{\lambda_\nu\}_{\nu=1}^\infty$ with $\lambda_\nu > 0$ and $\sum_{\nu=1}^\infty \lambda_\nu^2 < \infty$. $R(s, s')$ has the uniformly convergent Fourier expansion

$$R(s, s') = \sum_{\nu=1}^\infty \lambda_\nu \phi_\nu(s)\, \phi_\nu(s'). \tag{1.5}$$

Let $(\cdot, \cdot)$ be the inner product in $\mathscr{L}_2(S)$. For $f \in \mathscr{L}_2(S)$, we have

$$f = \sum_{\nu=1}^\infty f_\nu \phi_\nu, \qquad f_\nu = (f, \phi_\nu), \qquad \nu = 1, 2,\ldots,$$

$$Rf = \sum_{\nu=1}^\infty \lambda_\nu f_\nu \phi_\nu,$$

and we may define the symmetric square root $R^{-1/2}$ of R^{-1} by

$$R^{-1/2}f = \sum_{\nu=1}^\infty \frac{f_\nu}{(\lambda_\nu)^{1/2}} \phi_\nu, \tag{1.8}$$

for any $f \in \mathscr{L}_2(S)$ for which

$$\sum_{\nu=1}^\infty \frac{f_\nu^2}{\lambda_\nu} < \infty, \tag{1.9}$$

in which case,

$$\sum_{\nu=1}^\infty \frac{f_\nu^2}{\lambda_\nu} = \| R^{-1/2}f \|_{\mathscr{L}_2}^2.$$

Let

$$\mathscr{H}_R = \left\{g: g \in \mathscr{L}_2(S), \sum_{\nu=1}^\infty \frac{g_\nu^2}{\lambda_\nu} < \infty, g_\nu = (g, \phi_\nu)\right\},$$

and let R_s, for s fixed, be that function on S whose value at s' is given by

$$R_s(s') = R(s, s'). \tag{1.10}$$

The following facts about $\mathscr{H}_R$ may be verified by elementary methods, with the aid of (1.5):

(i) $\mathscr{H}_R$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_R$ given by

$$\langle f, g \rangle_R = \sum_{\nu=1}^\infty \frac{f_\nu g_\nu}{\lambda_\nu};$$

(ii) $R_s \in \mathscr{H}_R$, $\forall s \in S$;

(iii) $\langle R_s, z\rangle_R = z(s)$, $z \in \mathscr{H}_R$, $s \in S$.

Note that, for $z \in \mathscr{H}_R$,

$$\| R^{-1/2} z \|_{\mathscr{L}_2} = \| z \|_R, \tag{1.11}$$

where $\| \cdot \|_R$ is the norm in $\mathscr{H}_R$.

Properties (ii) and (iii) show that $\mathscr{H}_R$ is a reproducing kernel Hilbert space, (RKHS) with reproducing kernel $R(s, s')$. See, for example, Aronszajn [2], Yosida [15], Parzen [6], and Kimeldorf and Wahba [5] for further discussion of RKHS's and their uses.

The linear functional which, for fixed $s_* \in S$ maps $z \in \mathscr{H}_R$ into $z(s_*)$ is continuous in $\mathscr{H}_R$, as a consequence of (iii) and the Riesz representation theorem. Conversely, if $\mathscr{H}$ is any Hilbert space of functions for which the linear functionals,

$$z \to z(s_*), \tag{1.13}$$

are continuous for every $s_* \in S$, it is known that there exists a unique symmetric positive definite kernel $R(s, s')$ satisfying (ii) and (iii). To see this, note that, by the Riesz representation theorem, there exists $\xi_{s*} \in \mathscr{H}$ with the property

$$\langle \xi_{s*}, z\rangle = z(s_*). \tag{1.14}$$

Then, let

$$R(s, s') = \langle \xi_s, \xi_{s'}\rangle. \tag{1.15}$$

$R(s, s')$ of (1.15) is not required to be a continuous function of s and s' on $S \times S$. In this note, however, we always assume continuity. This assumption entails that $\mathscr{H}_R \subset C^0(S)$, where $C^0(S)$ is the continuous functions on S, by the inequalities

$$\begin{aligned} | z(s) - z(s+\epsilon)| &= |\langle z, R_s - R_{s+\epsilon}\rangle_R | \leqslant \| z \|_R \| R_s - R_{s+\epsilon} \|_R \\ &= \| z \|_R (R(s, s) - 2R(s, s+\epsilon) + R(s+\epsilon, s+\epsilon))^{1/2}. \end{aligned}$$

As an example of an RKHS, let L_m be an mth order linear differential operator with an m dimensional null space. Let $G_m(s, u)$ be the Green's function for the problem

$$L_m f = g \tag{1.16a}$$

$$f^{(\nu)}(0) = 0, \qquad \nu = 0, 1, 2, \ldots, m-1, \tag{1.16b}$$

and let $R(s, s')$ be given by

$$R(s, s') = \int_0^1 G_m(s, u)\, G_m(s', u)\, du. \tag{1.17}$$

Then,

$$\mathscr{H}_R = \{f: f^{(\nu)}(0) = 0,\ \nu = 0, 1,..., m-1,$$
$$f^{(m-1)} \text{ absolutely continuous, } L_m f \in \mathscr{L}_2[0, 1]\}$$

and the inner product in $\mathscr{H}_R$ is given by

$$\langle f, g\rangle_R = \int_0^1 (L_m f)(u)(L_m g)(u)\, du. \tag{1.18}$$

Examples of RKHS's, where boundary conditions such as (1.16b) are not imposed, may be found in [5], see also [6; 14] for further examples.

If Λ is a continuous linear functional on an RKHS with kernel $R(s, s')$, then

$$\Lambda z = \langle \eta, z\rangle_R ,$$

where η is given by

$$\eta(s) = \langle \eta, R_s\rangle_R = \Lambda R_s .$$

Thus, knowledge of the reproducing kernel $R(s, s')$ for $\mathscr{H}_R$ allows the explicit construction of the representer of any continuous linear functional.

Returning to the discussion of (1.1), we suppose R has been selected, and that K has the property that the family of linear functionals $\{\Lambda_t, t \in T\}$ defined by

$$\Lambda_t z = \int_S K(t, s)\, z(s)\, ds, \qquad t \in T, \quad z \in \mathscr{H}_R$$

are all continuous in $\mathscr{H}_R$, and linearly independent.

Then, (1.2) may be rewritten

$$u_i = \langle \eta_{t_i}, \hat{z}\rangle_R , \qquad i = 1, 2,..., n, \tag{1.19}$$

where $\eta_{t_i} \in \mathscr{H}_R$ is defined by

$$\eta_t(s) = \int_S K(t, u)\, R(s, u)\, du \qquad s \in S, \tag{1.20}$$

with $t = t_i$.

The previous assumption of linear independence is plausible, since otherwise there would exist constants $\{c_i\}_{i=1}^n$, with

$$\sum_{i=1}^n c_i \eta_{t_i} = 0,$$

and then

$$\left\langle \sum_{i=1}^{n} c_i \eta_{t_i}, z \right\rangle_R = \sum_{i=1}^{n} c_i \int_S K(t_i, s)\, z(s)\, ds = 0,$$

for every $z \in \mathscr{H}_R$.

Let V_n be the (n-dimensional) subspace of $\mathscr{H}_R$ spanned by $\{\eta_t, t \in \Delta\}$, and let P_{V_n} be the orthogonal projection in $\mathscr{H}_R$ onto V_n. If z is an arbitrary element in $\mathscr{H}_R$ satisfying (1.19), then $\hat{z} = P_{V_n} z$ satisfies (1.19) also and minimizes the norm $\| z \|_R$ among all such solutions. The element $P_{V_n} z$, being an element of V_n, can be solved for explicitly from (1.19), and is given by

$$(P_{V_n} z)(s) = (\eta_{t_1}(s), \eta_{t_2}(s), \ldots, \eta_{t_n}(s)\; Q_n^{-1}(u_1, u_2, \ldots, u_n)', \tag{1.21}$$

where Q_n is the $n \times n$ matrix with i, jth entry given by

$$\langle \eta_{t_i}, \eta_{t_j} \rangle_R = Q(t_i, t_j), \tag{1.22a}$$

$$Q(t, t') = \int_S \int_S K(t, s)\, R(s, s')\, K(t', s')\, ds\, ds'. \tag{1.22b}$$

Q_n is nonsingular by the presumed linear independence of the $\{\eta_t, t \in \Delta\}$.

Let $\mathscr{N}(K)$ be the null space of K in $\mathscr{H}_R$ (possibly the 0 element) and let $V = \mathscr{N}^{\perp}(K)$ (in $\mathscr{H}_R$). Then, by definition,

$$0 = \int_S K(t, s)\, z(s)\, ds, \qquad t \in T, \quad z \in \mathscr{H}_R \Rightarrow z \in \mathscr{N}(K). \tag{1.23}$$

Equation (1.23) may be written

$$0 = \langle \eta_t, z \rangle_R, \qquad t \in T, \qquad z \in \mathscr{H}_R \Rightarrow z \in V^{\perp}. \tag{1.24}$$

Thus, $\{\eta_t, t \in T\}$ span V.

If $R(s, s')$ is continuous on $S \times S$, then $\mathscr{H}_R$ is separable. Suppose that $Q(t, t') = \langle \eta_t, \eta_{t'} \rangle_R$ is continuous for $(t, t') \in T \times T$, then $\{\eta_t, t$ rational, $t \in T\}$ is dense in the set $\{\eta_t, t \in T\}$. Let P_V be the projection operator in $\mathscr{H}_R$ onto V, and let

$$\| \Delta \| = \max_i (t_{i+1} - t_i). \tag{1.25}$$

It then follows that

$$\lim_{\|\Delta\| \to 0} \| P_V z - P_{V_n} z \|_R = 0, \tag{1.26}$$

for any fixed $z \in \mathscr{H}_R$. (Obviously we possess no information concerning

$z - P_V z \in \mathcal{N}(K)$.) It appears that no rate holding uniformly for $z \in \mathcal{H}_R$ can be found for (1.26). In this note we investigate the convergence rates of $\| P_V z - P_{V_n} z \|_R$ and $|(P_V z)(s) - (P_{V_n} z)(s)|$ for $P_V z$ in a certain dense subset of $\mathcal{H}_R$.

To study $|(P_V z)(s) - (P_{V_n} z)(s)|$, it will be convenient to use the fact that

$$\begin{aligned} |(P_V z)(s) - (P_{V_n} z)(s)| &= |\langle P_V z - P_{V_n} z, P_V R_s - P_{V_n} R_s \rangle_R | \\ &\leqslant \| P_V z - P_{V_n} z \|_R \| P_V R_s - P_{V_n} R_s \|_R \\ &\leqslant \| P_V z - P_{V_n} z \|_R \| R_s - P_{V_n} R_s \|_R . \qquad (1.27) \end{aligned}$$

The approximate solution (1.21) is not, in general, the most appealing for computational work, since Q_n will become poorly conditioned as $(t_{i+1} - t_i) \to 0$ if $Q(t, t')$ is smooth. We study properties of the approximation (1.21), however, because of its close relationship to the approximate solution given by the method of regularization (see Eq. 1.28), which has been discussed and also investigated numerically by a number of authors. See Phillips [7], Tihonov [11, 12], Tihonov and Glasko [13], Ribiere [9], Strand and Westwater [10], Wahba [14], and Hunt [3]. However, there seems to be a lack of general theoretical results concerning the rate of convergence of these approximate solutions.

The approximate solutions given by the methods discussed by the authors above are (except for discretization) equivalent to finding $z \in \mathcal{H}_R$ to minimize

$$\sum_{i=1}^{n} (u_i - \langle \eta_{t_i}, z \rangle_R)^2 + \lambda \| z \|_R^2 , \qquad (1.28)$$

where λ is a parameter to be chosen. The solution $\tilde{z}$ to this problem is given by

$$\tilde{z}(s) = (\eta_{t_1}(s), \eta_{t_2}(s), \ldots, \eta_{t_n}(s))(Q_n + \lambda I)^{-1} (u_1, u_2, \ldots, u_n)'. \qquad (1.29)$$

In [7; 11; 12], $\| z \|_R^2$ is defined by some special case of the example of (1.18), with good numerical results presented for $m = 2$. There doesn't seem to be any obvious guideline for the choice of R, other than the observation that one would like the $\mathcal{H}_R$ norm of the unknown solution to be small.

We will use the notations K, K^*, and Q for the operators defined by

$$\begin{aligned} (Kf)(t) &= \int_S K(t, s) f(s)\, ds, && t \in T, \quad f \in \mathcal{L}_2(S), \\ (K^* f)(s) &= \int_T K(t, s) f(t)\, dt, && s \in S, \quad f \in \mathcal{L}_2(T), \qquad (1.30) \\ (Qf)(t) &= \int_T Q(t, t') f(t')\, dt', && t \in T, \quad f \in \mathcal{L}_2(T). \end{aligned}$$

Note that $Q = KRK^*$.

The main Theorem of this paper follows.

THEOREM. *Let $z \in \mathscr{H}_R$ have the property that*

$$P_V z \in RK^*(\mathscr{L}_2(T)), \tag{1.31}$$

or, equivalently,

$$u = Kz \in KRK^*(\mathscr{L}_2(T)) = Q(\mathscr{L}_2(T)),$$

and suppose that $Q(t, t')$ satisfies

(i) $(\partial^l/\partial t^l)\, Q(t, t')$ *exists and is continuous on $T \times T$ for $t \neq t'$,* (1.32)

$l = 0, 1, 2, \ldots, 2m$, $(\partial^l/\partial t^l)\, Q(t, t')$ *exists and is continuous on $T \times T$ for* $l = 0, 1, 2, \ldots, 2m - 2$;

(ii) $\lim_{t \uparrow t'} (\partial^{2m-1}/\partial t^{2m-1})\, Q(t, t')$ *and* $\lim_{t \downarrow t'} (\partial^{2m-1}/\partial t^{2m-1})\, Q(t, t')$ (1.33) *exist and are bounded for all $t' \in T$.*

Then,

$$\| P_V z - P_{V_n} z \|_R = O(\| \Delta \|^m). \tag{1.34}$$

Using (1.20), it is seen that (1.31) is equivalent to

$$(P_V z)(s) = \int_T \eta_{t'}(s)\, \rho(t')\, dt', \tag{1.35}$$

for some $\rho \in \mathscr{L}_2(T)$. It will be shown later (Lemma 2 et. seq.) that, if $Q(t, t')$ is continuous, then $RK^*(\mathscr{L}_2(T)) \subset V$ and is dense in V in the $\mathscr{H}_R$ norm.

An obvious and useful corollary follows.

COROLLARY. *Let $z \in \mathscr{H}_R$, $\psi \in RK^*(\mathscr{L}_2(T))$, then*

$$\begin{aligned} |\langle z, \psi\rangle_R - \langle P_{V_n} z, \psi\rangle_R| &= |\langle z - P_{V_n} z, \psi - P_{V_n}\psi\rangle_R| \\ &\leqslant \| z \|_R \| \psi - P_{V_n}\psi \|_R = O(\| \Delta \|^m). \end{aligned} \tag{1.36}$$

As an example of the application of this Corollary, suppose it is desired to approximate $\langle z, \psi\rangle_R$ for given ψ, knowing $\{u(t_i)\}_{i=1}^n$. The approximation may be taken as $\langle P_{V_n} z, \psi\rangle_R$, and the convergence rate of (1.36) for the approximation to $\langle z, \psi\rangle_R$, then obtains irrespective of conditions on z.

A useful special case is

$$\langle z, \psi\rangle_R = \int_S w(s)\, z(s)\, ds, \tag{1.37}$$

for which

$$\psi(s) = \int_S R(s, s')\, w(s')\, ds' = (Rw)(s). \tag{1.38}$$

If only $w \in K^*(\mathscr{L}_2(T))$, then $\psi \in RK^*(\mathscr{L}_2(T))$.

Section 2 is devoted to the proof of the theorem and associated lemmas. In certain very special examples the rate of convergence of $\| R_s - P_{V_n} R_s \|_R$ in (1.27) may also be found. Section 3 is given over to an example. The result there is equivalent to well known results in the convergence of derivatives of spline function approximations. It appears, however, that further results along this line depend rather delicately on detailed properties of K and R.

2. Proof of the Main Theorem

It will be convenient to define an auxiliary Hilbert space $\mathscr{H}_Q$. We let $\mathscr{H}_Q$ be the reproducing kernel Hilbert space with reproducing kernel $Q(t, t')$, $t, t' \in T$ defined by (1.22b), and inner product $\langle \cdot, \cdot \rangle_Q$. Let Q_t be that element of $\mathscr{H}_Q$ whose value at t' is given by

$$Q_t(t') = Q(t, t'), \tag{2.1}$$

and let $\mathscr{H}_{T_n}$ be the subspace of $\mathscr{H}_Q$ spanned by the elements

$$\{Q_{t_i}\}_{i=1}^n .$$

Let P_{T_n} be the projection operator in $\mathscr{H}_Q$ onto $\mathscr{H}_{T_n}$.

Lemma 1. *Given $z \in \mathscr{H}_R$ let u be defined by*

$$u(t) = \langle \eta_t , z \rangle_R , \qquad t \in T. \tag{2.2}$$

Then $u \in \mathscr{H}_Q$ and

$$\| P_V z - P_{V_n} z \|_R = \| u - P_{T_n} u \|_Q .$$

Proof. Since

$$\langle Q_t , Q_{t'} \rangle_Q = Q(t, t') = \langle \eta_t , \eta_{t'} \rangle_R , \qquad t, t' \in T \tag{2.3}$$

and $\{Q_t , t \in T\}$ span $\mathscr{H}_Q$, there is an isometric isomorphism between $\mathscr{H}_Q$ and V generated by the correspondence "$\sim$",

$$Q_t \in \mathscr{H}_Q \sim \eta_t \in V, \qquad t \in T. \tag{2.4}$$

Obviously,

$$\mathscr{H}_{T_n} \sim V_n ,$$

under this isomorphism. Furthermore, since for $z \in \mathscr{H}_R$,

$$\langle \eta_t , z \rangle_R = \langle \eta_t , P_V z \rangle_R = u(t) = \langle Q_t , u \rangle_Q , \tag{2.5}$$

we have

$$P_V z \sim u, \tag{2.6}$$

and

$$P_{V_n} z \sim P_{T_n} u, \tag{2.7}$$

and, hence,

$$\| P_V z - P_{V_n} z \|_R = \| u - P_{T_n} u \|_Q . \tag{2.8}$$

This completes the proof of Lemma 1.

LEMMA 2. *Suppose z has a representation*

$$z(s) = \int_T \eta_{t'}(s)\, \rho(t')\, dt', \tag{2.9}$$

for some $\rho \in \mathscr{L}_2(T)$, where $\eta_t(s)$ is defined by (1.20).[1] *Then $z \in V$ and $z \sim u$ under the correspondence "$\sim$" of* (2.4), *where*

$$u(t) = \int_T Q_t(t')\, \rho(t')\, dt'. \tag{2.10}$$

Proof. It is sufficient to prove the result for ρ continuous, as follows. Suppose $\{\rho_\nu\}_{\nu=1}^{\infty}$ is a sequence of continuous functions converging (in the $\mathscr{L}_2$ norm) to $\rho \in \mathscr{L}_2$. Then, u_ν , given by

$$u_\nu(t) = \int_T Q_t(t')\, \rho_\nu(t')\, dt',$$

is in $\mathscr{H}_Q$, and corresponds to z_ν , given by

$$z_\nu(s) = \int_T \eta_{t'}(s)\, \rho_\nu(t')\, dt'.$$

Then, u_ν converges pointwise uniformly and, hence, strongly in $\mathscr{H}_Q$ to u. Similarly z_ν converges strongly in V to z and $z \sim u$.

[1] That is, $z = RK^*\rho$, $\rho \in \mathscr{L}_2(T)$.

Now, let ρ be continuous and let $\Pi_l = \{t_{1l}, t_{2l}, \ldots, t_{ll}\}$, $l = 1, 2,\ldots$ be a sequence of partitions of T, such that, for every t, the Riemann sums for Π_l for the integral

$$\int_T Q(t, t')\, \rho(t')\, dt' \tag{2.11}$$

converge.

Then $z_{(l)}$, $l = 1, 2,\ldots$ defined by

$$z_{(l)}(s) = \sum_{j=1}^{l-1} \eta_{t_j}(s)\, \rho(t_{jl})(t_{j+1,l} - t_{jl}), \qquad l = 1, 2,\ldots \tag{2.12}$$

is a Cauchy sequence of elements in V, which converge pointwise to $z(s)$ of (2.9) and $u_{(l)}$, $l = 1, 2,\ldots$ defined by

$$u_{(l)}(t) = \langle \eta_t, z_{(l)} \rangle_R = \left\langle \eta_t, \sum_{j=1}^{l-1} \eta_{t_{jl}} \rho(t_{jl})(t_{j+1,l} - t_{jl}) \right\rangle_R \quad l = 1, 2,\ldots \tag{2.13}$$

is a Cauchy sequence of elements in $\mathscr{H}_Q$ which converge pointwise to $u(t)$ given by (2.10). But by (2.5) $u_{(l)} \sim z_{(l)}$ so we must have $u \sim z$ with u and z defined by (2.9) and (2.10), thus, completeing the proof of Lemma 2.

Incidentally, Lemmas 1 and 2 may be used to show that $RK^*(\mathscr{L}_2(T))$ is dense in V (in the $\mathscr{H}_R$-norm). To see this, note that it is only necessary to show, that, for each $t_* \in T$, and every $\epsilon > 0$, there exists $\rho^\epsilon \in \mathscr{L}_2(T)$ such that $x^\epsilon = RK^*\rho^\epsilon$ satisfies

$$\| \eta_{t_*} - x^\epsilon \|_R \leqslant \epsilon. \tag{2.14}$$

Now

$$x^\epsilon(s) = (RK^*\rho^\epsilon)(s) = \int_T \eta_t(s)\, \rho^\epsilon(t)\, dt \tag{2.15}$$

and so, by Lemmas 1 and 2,

$$\| \eta_{t_*} - x^\epsilon \|_R = \| Q_{t^*} - y^\epsilon \|_Q, \tag{2.16}$$

where

$$y^\epsilon(t) = \int_T Q_t(t')\, \rho^\epsilon(t')\, dt'. \tag{2.17}$$

But

$$\| Q_{t^*} - y^\epsilon \|_Q^2 = Q(t_*, t_*) - 2 \int_T Q(t_*, t')\, \rho^\epsilon(t')\, dt' + \int_T \int_T \rho^\epsilon(t)\, Q(t, t')\, \rho^\epsilon(t')\, dt\, dt', \tag{2.18}$$

and it is obvious that if $Q(t, t')$ is continuous on $T \times T$, then there exists a $\rho^\epsilon \in \mathscr{L}_2(T)$ such that the right hand side of (2.18) is $\leqslant \epsilon^2$.

We now proceed to the prove the theorem.

Proof of theorem. By Lemmas 1 and 2 it is sufficient to prove

$$\| u - P_{T_n} u \|_Q = O(\| \Delta \|^m), \tag{2.19}$$

where

$$u(t) = \int_T Q_t(t') \rho(t') \, dt'.$$

We actually show that

$$\| u - P_{T_n} u \|_Q \leqslant (6m)^m \, (C_1(t_n - t_1) + C_2)^{1/2} \left[\int_T \rho^2(t) \, dt \right]^{1/2} \| \Delta \|^m, \tag{2.20a}$$

where

$$C_1 = (1 + 2m\Theta_m) \sup_{\substack{\xi \neq \xi' \\ \xi, \xi' \in T}} \left| \frac{1}{(2m)!} \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, \xi') \right|, \tag{2.20b}$$

$$C_2 = 2(1 + 2m\Theta_m) \sup_{\xi, \xi' \in T} \left| \frac{1}{(2m-1)!} \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, \xi') \right|, \tag{2.20c}$$

$$\Theta_m = [3(2m - 1)]^{2m-1}, \tag{2.20d}$$

and it is understood that if $(\partial^{2m-1}/\partial \xi^{2m-1}) \, Q(\xi, \xi')$ is undefined the maximum of the left and right absolute derivative is taken.

If $\tilde{u}$ is any element in $\mathscr{H}_Q$ of the form

$$\tilde{u} = \sum_i Q_{t_i} \int_T c_i(t) \rho(t) \, dt, \tag{2.21}$$

then, since $\tilde{u} \in \mathscr{H}_{T_n}$, we have

$$\| u - P_{T_n} u \|_Q^2 \leqslant \| u - \tilde{u} \|_Q^2 . \tag{2.22}$$

The proof proceeds by finding a set of functions $\{c_i(t)\}_{i=1}^n$ so that $\| u - \tilde{u} \|_Q^2$ with $\tilde{u}$ defined by (2.21) is bounded by the right hand side of (2.20a).

Without loss of generality we assume that

$$\frac{\max_i (t_{i+1} - t_i)}{\min_j (t_{j+1} - t_j)} \leqslant 3. \tag{2.23}$$

No generality is lost, because we may delete elements from Δ without reducing the right hand side of (2.22). From any set Δ with mesh norm $\| \Delta \|$ we can always choose a subset Δ' with property (2.23), and

$$\| \Delta' \| \leqslant 3 \| \Delta \| \tag{2.24}$$

by dividing the interval T into successive subintervals of length $\| \Delta \|$ and selecting exactly one t_i from every other subinterval. We assume this has been done and the set Δ', which we will relabel $\Delta = \{t_1, t_2, ..., t_n\}$, has mesh norm bounded by $3 \| \Delta \|$.

Now, since u satisfying (2.10) satisfies

$$\langle u, u \rangle_Q = \int_T \int_T \rho(t)\, Q(t, t')\, \rho(t')\, dt\, dt', \tag{2.25}$$

$$\langle u, Q_{t_i} \rangle_Q = u(t_i) = \int_T Q_{t_i}(t)\, \rho(t)\, dt, \tag{2.26}$$

it follows that

$$\| u - \tilde{u} \|_Q^2$$
$$= \int_T \int_T \rho(t) \rho(t') \left\langle Q_t - \sum_{i=1}^n c_i(t)\, Q_{t_i}, Q_{t'} - \sum_{j=1}^n c_j(t')\, Q_{t_j} \right\rangle_Q dt\, dt'. \tag{2.27}$$

Without loss of generality suppose $n = N(2m - 1) + 1$ for some integer N. To simplify the notation, let

$$t_{k,i} = t_{k(2m-1)+i}, \qquad k = 0, 1, 2,..., N - 1, \quad i = 1, 2,..., 2m.$$

(Note that $t_{k,2m} = t_{k+1,1}$). Let I_k be the interval

$$I_k = (t_{k,1}, t_{k+1,1}], \qquad k = 0, 1, 2,..., N - 1.$$

For $t \in I_k$, we will approximate Q_t by that linear combination of

$$\{Q_{t_{k,i}}\}_{i=1}^{2m},$$

which corresponds to Lagrange (polynomial) interpolation of degree $2m - 1$. More precisely, let

$$\begin{aligned} p_{k,i}(t) &= \prod_{\substack{\nu=1 \\ \nu \neq i}}^{2m} (t - t_{k,\nu}) \Big/ \prod_{\substack{\nu=1 \\ \nu \neq i}}^{2m} (t_{k,i} - t_{k,\nu}), \qquad && t \in I_k, \\ &= 0, && t \notin I_k, \end{aligned} \tag{2.28}$$
$$k = 0, 1, 2,..., N - 1,$$
$$i = 1, 2,..., 2m.$$

For $t \in I_k$, $t' \in I_l$, set

$$\left\langle Q_t - \sum_{i=1}^{n} c_i(t)\, Q_{t_i}\,,\, Q_{t'} - \sum_{j=1}^{n} c_j(t')\, Q_{t_j} \right\rangle_Q$$

$$= \left\langle Q_t - \sum_{i=1}^{2m} p_{k,i}(t)\, Q_{t_{k,i}}\,,\, Q_{t'} - \sum_{j=1}^{2m} p_{l,j}(t')\, Q_{t_{l,j}} \right\rangle_Q$$

$$= Q_{t'}(t) - \sum_{j=1}^{2m} p_{l,j}(t')\, Q_{t_{l,j}}(t)$$

$$- \sum_{i=1}^{2m} p_{k,i}(t) \left\{ Q_{t'}(t_{k,i}) - \sum_{j=1}^{2m} p_{l,j}(t')\, Q_{t_{l,j}}(t_{k,i}) \right\}. \tag{2.29}$$

We want to use the Newton form of the remainder for Lagrange interpolation, [4, p. 248]. For any $f(t)$, $t \in I_k$,

$$f(t) - \sum_{i=1}^{2m} p_{k,i}(t) f(t_{k,i}) = \prod_{i=1}^{2m} (t - t_{k,i}) f[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t], \tag{2.30}$$

where $f[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t]$ is the $2m$th order divided difference.

Using (2.30) with $f(t) = Q_{t'}(t) - \sum_{j=1}^{2m} p_{l,j}(t')\, Q_{t_{l,j}}(t)$, the right hand side (r.h.s.) of (2.29) is seen to be equal to

$$\text{r.h.s. } (2.29) = \prod_{i=1}^{2m} (t - t_{k,i}) \Big\{ Q_{t'}[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t]$$

$$- \sum_{j=1}^{2m} p_{l,j}(t')\, Q_{t_{l,j}}[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t] \Big\}. \tag{2.31}$$

For any f, we know that if f has $2m$ continuous derivatives in I_k, then

$$f[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t] = \frac{f^{(2m)}(\xi)}{(2m)!}, \tag{2.32}$$

for some $\xi \in I_k$. If we only know that $f^{(2m-1)}(t)$ is continuous except for a finite number of finite jumps, then we may write the $2m$th order divided difference as a divided difference of two $2m - 1$-st order divided differences,

$$f[t_{k,1}, t_{k,2}, \ldots, t_{k,2m}, t] = \frac{1}{(t_{k,2m} - t_{k,1})} \{ f[t_{k,1}, t_{k,2}, \ldots, t_{k,2m-1}, t]$$

$$- f[t_{k,2}, t_{k,3}, \ldots, t_{k,2m}, t] \}, \tag{2.33}$$

and know that the term in brackets in (2.33) is bounded in absolute value by $2 \sup_{t \in I_k} |[1/(2m-1)!] f^{(2m-1)}(t)|$.

By (2.23) and (2.28),

$$|p_{l,j}(t)| \leqslant \Theta_m, \qquad \Theta_m = [3(2m-1)]^{2m-1}. \tag{2.34}$$

Therefore, for $t \in I_k$, $t' \in I_l$, $k \neq l$, we use (2.32) and have the following bound on the right hand side of (2.31):

$$\begin{aligned} |\text{ r.h.s. } (2.29)| &= |\text{ r.h.s. } (2.31)| \\ &\leqslant (t_{k,2m} - t_{k,1})^{2m} (1 + 2m\Theta_m) \sup_{\substack{\xi \in I_k \\ \xi' \in I_l}} \left| \frac{1}{(2m)!} \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, \xi') \right| \\ &\leqslant (t_{k,2m} - t_{k,1})^{2m} C_1, \qquad t \in I_k, \quad t' \in I_l, \quad k \neq l \end{aligned} \tag{2.35}$$

where C_1 is given by (2.20b).

For $t, t' \in I_k$, we use (2.33) and have the following bound:

$$\begin{aligned} |\text{ r.h.s. } (2.29)| &= |\text{ r.h.s. } (2.31)| \\ &\leqslant (t_{k,2m} - t_{k,1})^{2m-1} 2(1 + 2m\Theta_m) \\ &\quad \times \sup_{\xi, \xi' \in I_k} \left| \frac{1}{(2m-1)!} \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, \xi') \right| \\ &\leqslant (t_{k,2m} - t_{k,1})^{2m-1} C_2, \qquad t, t' \in I_k, \end{aligned} \tag{2.36}$$

where C_2 is given by (2.20c) and where it is understood that if $(\partial^{2m-1}/\partial \xi^{2m-1}) Q(\xi, \xi')$ is undefined the maximum of the left and right absolute derivative is taken. Thus, using (2.27), (2.29), (2.31), (2.35), and (2.36),

$$\begin{aligned} \| u - \tilde{u} \|_Q^2 \leqslant C_1 &\sum_{\substack{k,l=0 \\ k \neq l}}^{N-1} (t_{k,2m} - t_{k,1})^{2m} \int_{I_k} \int_{I_l} |\rho(t)| \, |\rho(t')| \, dt \, dt' \\ &+ C_2 \sum_{k=1}^{N-1} (t_{k,2m} - t_{k,1})^{2m-1} \int_{I_k} \int_{I_k} |\rho(t)| \, |\rho(t')| \, dt \, dt'. \end{aligned} \tag{2.37}$$

Since

$$\int_{I_k} |\rho(t)| \, dt \leqslant (t_{k,2m} - t_{k,1})^{1/2} \left[\int_{I_k} \rho^2(t) \, dt \right]^{1/2}, \tag{2.38}$$

(2.37) becomes

$$\begin{aligned}
\| u - \tilde{u} \|_Q^2 &\leqslant C_1 \sum_{\substack{k,l=0\\ k\neq l}}^{N-1} (t_{k,2m} - t_{k,1})^{2m+1/2} (t_{l,2m} - t_{l,1})^{1/2} \left[\int_{I_k} \rho^2(t)\, dt\right]^{1/2} \\
&\quad \times \left[\int_{I_l} \rho^2(t)\, dt\right]^{1/2} + C_2 \sum_{k=0}^{N-1} (t_{k,2m} - t_{k,1})^{2m} \int_{I_k} \rho^2(t)\, dt. \\
&\leqslant [\sup_k (t_{k,2m} - t_{k,1})^{2m}] \Big\{ C_1 \sum_{\substack{k,l=0\\ k\neq l}}^{N-1} (t_{k,2m} - t_{k,1})^{1/2} \left[\int_{I_k} \rho^2(t)\, dt\right]^{1/2} \\
&\quad \times (t_{l,2m} - t_{l,1})^{1/2} \left[\int_{I_l} \rho^2(t)\, dt\right]^{1/2} + C_2 \sum_{k=0}^{N-1} \int_{I_k} \rho^2(t)\, dt \Big\} \\
&\leqslant \sup_k (t_{k,2m} - t_{k,1})^{2m} \times [C_1(t_n - t_1) + C_2] \int_T \rho^2(t)\, dt \\
&\leqslant (2m)^{2m} (3 \| \Delta \|)^{2m} (C_1(t_n - t_1) + C_2) \int_T \rho^2(t)\, dt. \qquad (2.39)
\end{aligned}$$

3. Behavior of $\| R_s - P_{V_n} R_s \|_R$

We consider only a special example here. It will appear from the discussion that a general theorem is unavailable without further detailed assumptions concerning $K(t, s)$.

Let $S = [0, 1]$, $T = [0, 1]$, and let

$$K(t, s) = \frac{(t - s)_+^{l-1}}{(l - 1)!} \qquad \begin{aligned} (u)_+ &= u, \quad u \geqslant 0 \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{3.1}$$

l some integer. Then, since K is the Green's function for the operator L, defined by

$$Lu = u^{(l)},$$

with boundary conditions $u^{(\nu)}(0) = 0$, $\nu = 0, 1, 2,..., l - 1$, we have

$$z = u^{(l)}, \tag{3.2}$$

as the solution of (1.1).

Let

$$R(s, s') = \int_0^1 \frac{(s-u)_+^{k-1}}{(k-1)!} \frac{(s'-u)_+^{k-1}}{(k-1)!} \, du. \tag{3.3}$$

It follows that

$$\eta_t(s) = \int_0^1 \frac{(t-u)_+^{m-1}}{(m-1)!} \frac{(s-u)_+^{k-1}}{(k-1)!} \, du, \tag{3.4}$$

$$Q(t, t') = \int_0^1 \frac{(t-u)_+^{m-1}}{(m-1)!} \frac{(t'-u)_+^{m-1}}{(m-1)!} \, du, \tag{3.5}$$

with $k + l = m$.

In this example, $\mathscr{H}_R = \{z: z^{(\nu)}(0) = 0,\ \nu = 0, 1, 2,..., k-1,\ z^{(k-1)}$ absolutely continuous, $z^{(k)} \in L_2[0, 1]\}$,

$$\langle z_1, z_2 \rangle_R = \int_0^1 z_1^{(k)}(s)\, z_2^{(k)}(s) \, ds, \tag{3.6}$$

and $\mathscr{H}_Q = \{u: u^{(\nu)}(0) = 0,\ \nu = 0, 1, 2,..., m-1,\ u^{(m-1)}$ absolutely continuous, $u^{(m)} \in L_2[0, 1]\}$, with

$$\langle u_1, u_2 \rangle_Q = \int_0^1 u_1^{(m)}(t)\, u_2^{(m)}(t) \, dt. \tag{3.7}$$

If, in general we view the operator K, defined by

$$(Kz)(t) = \int_T K(t, s)\, z(s) \, ds, \tag{3.8}$$

as an operator from V to $\mathscr{H}_Q$, it is 1:1 invertible and

$$u = Kz \Rightarrow K^{-1}(P_{T_n} u) = P_{V_n} z. \tag{3.9}$$

Returning to the example, the solution to the problem: Find $\hat{u} \in \mathscr{H}_Q$, satisfying[2]

$$u(t_i) = u_i, \qquad i = 1, 2,..., n \tag{3.10}$$

to minimize

$$\| u \|_Q^2 = \int_0^1 [u^{(m)}(t)]^2 \, dt,$$

[2] To avoid trivial difficulties, let $t_1 > 0$.

is given by

$$\hat{u}(t) = (Q_{t_1}(t), \ldots, Q_{t_n}(t))\, Q_n^{-1}(u_1, u_2, \ldots, u_n)'. \tag{3.11}$$

($u = P_{T_n} u$, where $u \in \mathscr{H}_Q$ is any element satisfying (3.10).)

The solution to this problem is well known to be the (unique) polynomial spline of interpolation to $u(t)$, $t \in \Delta$, of degree $2m - 1$ and continuity class C^{2m-2}, and satisfying the boundary conditions for elements of $\mathscr{H}_Q$. Here,

$$K^{-1} u = u^{(l)}, \tag{3.12}$$

and it is easily verified that

$$(d^l/ds^l)(P_{T_n} u)(s) = (P_{V_n} z)(s). \tag{3.13}$$

In this example, u having a representation of the form (2.10) implies that

$$\begin{aligned} u^{(2m)} &= \rho \\ u^{(\nu)}(0) = u^{(m+\nu)}(1) &= 0, \qquad \nu = 0, 1, 2, \ldots, m-1. \end{aligned} \tag{3.14}$$

Thus, $P_{V_n} z$ is the lth derivative of a polynomial spline function of degree $2m - 1$ interpolating to a function $u \in C^{2m}$. Such approximants are well known to have convergence rates $\leqslant O(\| \Delta \|^{2m-l-1/2})$, where $k + l = m$. See [1].

By (1.27) and the theorem, $\| R_s - P_{V_n} R_s \|_R = O(\| \Delta \|^{k-1/2})$ would insure the above result, namely

$$| z(s) - (P_{V_n} z)(s) | \leqslant O(\| \Delta \|^{m+k-1/2}). \tag{3.15}$$

A proof that $\| R_s - P_{V_n} R_s \|_R = O(\| \Delta \|^{k-1/2})$ for general K and R as in (3.3) might begin by writing

$$\begin{aligned} &\| R_s - P_{V_n} R_s \|_R^2 \\ &\quad \leqslant \left\| R_s - \sum_{i=1}^{n} d_i(s)\, \eta_{t_i} \right\|_R^2 \\ &\quad = \int_0^1 \left[\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{i=1}^{n} d_i(s) \int_0^1 K(t_i, v) \frac{(v-u)_+^{k-1}}{(k-1)!}\, dv \right]^2 du, \end{aligned} \tag{3.16}$$

where, for fixed s, the $\{d_i(s)\}$ are constants to be found.

To continue a proof, assume K is such that $K(t_i, v) = 0$ for $v > t_i$. Then, for $s \in [t_i, t_{i+m'}]$ we have, for any m',

$$| \text{r.h.s. } (3.16)|$$

$$\leqslant \int_0^{t_i} \left[\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{\nu=0}^{m'} d_{i+\nu}(s) \int_0^{t_{i+\nu}} K(t_{i+\nu}, v) \frac{(v-u)_+^{k-1}}{(k-1)!} dv\right]^2 du$$

$$+ \int_{t_i}^{t_{i+m'}} \left[\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{\nu=0}^{m'} d_{i+\nu}(s) \int_0^{t_{i+\nu}} K(t_{i+\nu}, v) \frac{(v-u)_+^{k-1}}{(k-1)!} dv\right]^2 du. \tag{3.17}$$

If $K(t, v) = [(t-v)_+^{l-1}/(l-1)!]$, then

$$\int_0^{t_j} K(t_j, v) \frac{(v-u)_+^{k-1}}{(k-1)!} dv = \frac{(t_j - u)_+^{m-1}}{(m-1)!}, \tag{3.18}$$

and the integrand in the first term on the right in (3.17) is the square of a polynomial of degree $m - 1$ in u. Set $m' = m - 1$ and let $q_{i,\nu}(s)$ be the polynomial of degree $m - 1$ with

$$q_{i,\nu}(t_{i+\theta}) = \begin{cases} 1, & \nu = \theta \\ 0, & \nu \neq \theta \end{cases} \qquad \nu, \theta = 0, 1, 2,..., m-1. \tag{3.19}$$

Letting

$$d_{i+\nu}(s) = \begin{cases} (d^l/ds^l)\, q_{i,\nu}(s), & \nu = 0, 1, 2,..., m-1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.20}$$

the integrand in the first term on the right hand side of (3.17) is then identically zero, and, assuming (2.23), it is not hard to prove that the second term on the right hand side of (3.17) is bounded by $D_m(t_{i+m-1} - t_i)^{2k-1}$, where D_m is a constant. Thus, $\| R_s - P_{V_n} R_s \|_R = O(\| \Delta \|^{k-1/2})$. This approach to a proof clearly breaks down in general, however, unless the polynomial $p(u) = (s-u)^{k-1}$, $u \in [0, t_i]$ is in the linear span of the $m' + 1$ functions of u on $[0, t_i]$,

$$\int_0^{t_{i+\nu}} K(t_{i+\nu}, v) \frac{(v-u)_+^{k-1}}{(k-1)!} dv, \qquad u \in [0, t_i], \quad \nu = 0, 1, 2,..., m'. \tag{3.21}$$

References

1. T. H. Ahlberg, E. W. Nilson, and J. L. Walsh, "The Theory of Splines and Their Applications," Academic Press, New York, 1967.
2. N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337–404.

3. B. R. HUNT, Statistical aspects of deconvolution, Report LA-4556-MS, Los Alamos Scientific Laboratory, Los Alamos, New Mexico.
4. E. ISAACSON AND H. KELLER, "Analysis of Numerical Methods," John Wiley & Sons, New York, 1966.
5. G. KIMELDORF AND G. WAHBA, Some results on Tchebycheffian spline functions, *Journ. Math. Anal. Appl.* **33** (1971), 82–95.
6. E. PARZEN, Statistical inference on time series by RKHS methods, Proceedings of the 12th Biennial Seminar of the Canadian Mathematical Congress (1970), R. Pyke, ed.
7. D. L. PHILLIPS, A technique for the numerical solution of certain integral equations of the first kind, *J. Assoc. Comput. Mach.* **9** (1962), 84–97.
8. F. RIESZ AND SZ.-NAGY, "Functional Analysis," Ungar, New York, 1955.
9. G. RIBIERE, Regularisation d'operateurs, *Revue Française d'Informatique et de Recherche Opérationnelle* **1** (1967), 57–79.
10. O. N. STRAND AND E. R. WESTWATER, Statistical estimation of the numerical solution of a Fredholm integral equation of the first kind, *J. Assoc. Comput. Mach.* **15** (1968), 100–114.
11. A. M. TIHONOV, Solution of incorrectly formulated problems and the regularization method, *Soviet Mathematics* **4** (1963), 1035–38.
12. A. N. TIHONOV, Regularization of incorrectly posed problems, *Soviet Math. Dokl.* **4** (1963), 1624–27.
13. A. N. TIHONOV AND V. B. GLASKO, An approximate solution of Fredholm integral equations of the first kind, *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** (1964), 564–571.
14. G. WAHBA, On the approximate solution of Fredholm integral equations of the first kind, University of Wisconsin Mathematics Research Center Technical Summary Report #990, May 1969.
15. K. YOSIDA, "Functional Analysis," second ed., Springer, New York, 1968.